# On Existence of Tolerance Stable Diffeomorphisms (Theory of Dynamical Systems and Its Applications)



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443巻 1981年 223-242



# ON EXISTENCE OF TOLERANCE STABLE DIFFEOMORFFISMS*

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## §1. Introduction

We consider a compact smooth manifold M. $\mathrm{Diff}^1(M)$ denotes the space of $C^1$-diffeomorphisms of M onto itself with the usual $C^1$-topology. In the research of the qualitative theory of dynamical systems there is a desire to find a concept of stability of geometric global structure of orbits such that this stable systems are dense in the space of dynamical systems on M. Structural stability does not satisfy the density condition in $\mathrm{Diff}^1(M)$. Tolerance stability (see §2 for definition) is a candidate for the density property [7, p.294]. Concerning tolerance stability there are researches as [6],[7],[8], and [2].

If $f \in \mathrm{Diff}^1(M)$ is structurally stable in strong sense, f is topologically stable in $\mathrm{Diff}^1(M)$ (see §2 for definition). Moreover, topological stability implies tolerance stability [A. Morimoto, 2]. The proof of this property will be introduced in §2.

The main result of this paper is to show the existence of diffeomorphisms on any compact manifold M which are tolerance stable but not topologically stable in $\mathrm{Diff}^1(M)$, so that, not structurally stable in strong sence. This will be proved in §§3,4 and 5.

## §2. Definitions and statement of results.

We denote by Homeo(M) the set of homeomorphisms of M onto itself; the topology on Homeo(M) is given by the neighborhood $N_\varepsilon(f)$ of $f \in \mathrm{Homeo}(M)$

* The auther is partly supported by Grant in Aid for Scientific Research Project No. 546004.

$$N_{\varepsilon}(f) = \{ g ; d(f,g) < \varepsilon \}, \quad \varepsilon > 0 .$$

Here, for a metric d on M, $d(f,g)<\varepsilon$ means

$$d(f(x),g(x)) < \varepsilon \quad \text{for} \quad x \in M .$$

To state the definition of tolerance stability, we need the following definition :

Definition (2,1). $f,g \in \mathrm{Homeo}(M)$ are <u>orbit</u>-$\varepsilon$-<u>equivalent</u>, $\varepsilon > 0$, if

1. for every f-orbit $O_f$, there is a g-orbit $O_g$ such that
   (a) $O_f \subset U_{\varepsilon}(O_g)$
   (b) $O_g \subset U_{\varepsilon}(O_f)$, and
2. for every g-orbit $O_g$, there is a f-orbit $O_f$ such that
   (a) $O_g \subset U_{\varepsilon}(O_f)$
   (b) $O_f \subset U_{\varepsilon}(O_g)$.

Here, $U_{\varepsilon}(*)$ is the $\varepsilon$-neighborhood of $*$.

Suppose that a subset $\mathcal{D}$ of Homeo(M) is given a topology not coaser than that of Homeo(M).

Definition (2.2). An element $f \in \mathcal{D}$ is <u>tolerance-stable</u> in $\mathcal{D}$ if for every $\varepsilon > 0$ there is a neighborhood N of f in $\mathcal{D}$ (with respect to the given topology on $\mathcal{D}$) such that, for every $g \in N$, f and g are orbit-$\varepsilon$-equivalent.

Definition (2.3). An element $f \in \mathcal{D}$ is <u>topologically stable</u> in $\mathcal{D}$, if for any $\varepsilon > 0$ there is a neighborhood N of f in $\mathcal{D}$ such that for every $g \in N$ there is a continuous map $h : M \to M$ satisfying

(a) $d(h, i_M) < \varepsilon$ , where $i_M$ is the identity map of M,

(b) $hg = fh$.

The following property is mentioned and proved by A.Morimoto in [2]. We introduce this :

Proposition. If M is a compact topological manifold and $f \in \mathrm{Homeo}(M)$ is topologically stable in $\mathscr{D}$ then f is tolerance stable in $\mathscr{D}$, for any subset $\mathscr{D} \subset \mathrm{Homeo}(M)$.

Proof. For closed non-empty subsets A and B of M, let

$$\bar{d}(A,B) = \max\{\max_{a\in A} d(a,B),\ \max_{b\in B} d(b,A)\} ,$$

where, $d(a,B) = \min_{b\in B} d(a,b)$. $O_f(x)$ denotes the f-orbit of x ; $O_f(x) = \{f^i(x) ; i \in \mathbb{Z}\}$. Put $\bar{O}_f(x) = \mathrm{Cl}(O_f(x))$. By the assumption, for every $\varepsilon > 0$, there is a neighborhood N of f in $\mathscr{D}$ such that for every $g \in N$ there is $h : M \to M$ satisfying (a) and (b) in Definition (2.3). By (b), $h(O_g(x)) = O_f(h(x))$ for every $x \in M$. Hence,

$$\bar{d}(\bar{O}_g(x), \bar{O}_f(h(x))) = \bar{d}(\bar{O}_g(x), h(\bar{O}_g(x))) < \varepsilon .$$

Therefore, for any g-orbit $O_g$ there is f-orbit $O_f$ such that $O_g \subset U_{2\varepsilon}(O_f)$ and $O_f \subset U_{2\varepsilon}(O_g)$. Since M is a compact manifold, We can prove that $d(h, i_M) < \varepsilon$ implies that $h : M \to M$ is a surjection if $\varepsilon > 0$ is sufficiently small. We may assume that $\varepsilon$ is taken so small that this property is satisfied. Hence for every $x \in M$ there is $y \in M$ such that $h(y) = x$. Then

$$\begin{aligned}\bar{d}(\bar{O}_f(x), \bar{O}_g(y)) &= \bar{d}(\bar{O}_f(h(y)), \bar{O}_g(y)) \\ &= \bar{d}(h(\bar{O}_g(y)), \bar{O}_g(y)) < \varepsilon .\end{aligned}$$

Hence, for any f-orbit $O_f$ there is g-orbit $O_g$ such that $O_f \subset U_{2\varepsilon}(O_g)$ and $O_g \subset U_{2\varepsilon}(O_f)$. Therefore, f is tolerance stable in $\mathcal{D}$.

Definition (2.4). Two elements $f, g \in \mathrm{Diff}^1(M)$ are <u>topologically</u> $\varepsilon$-<u>conjugate</u> if there is a homeomorphism $h : M \to M$ such that $hg = fh$ and $d(h(x), x) < \varepsilon$ for every $x \in M$. f,g are <u>topologically conjugate</u> if there is a homeomorphism h such that $hg = fh$.

Definition (2.5). An element $f \in \mathrm{Diff}^1(M)$ is <u>structurally stable in strong sense</u> if for every $\varepsilon > 0$ there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ such that every $g \in N$ are topologically $\varepsilon$-conjugate to f. f is <u>structurally stable</u>, if there is N such that, for every $g \in N$, f and g are topologically conjugate.

Structural stability in strong sense implies structural stability and topological stability in $\mathrm{Diff}^1(M)$. If $f \in \mathrm{Diff}^1(M)$ satisfies Axiom A and strong transversality condition then f is structurally stable in strong sense [4].

Theorem. Let M be a compact differentiable manifold. There is a diffeomorphism f, in the boundary $\partial\Sigma$ of the set $\Sigma$ of all structurally stable elements in $\mathrm{Diff}^1(M)$, such that

(a) f is tolerance-stable in $\mathrm{Diff}^1(M)$, and

(b) f is not topologically stable in $\mathrm{Diff}^1(M)$, so that, f is not structurally stable in strong sense.

§3. Construction of f.

Theorem is proved in §§3,4 and 5. In these sections M is assumed to have $\dim M \geqq 2$. But to the readers of these sections the proof of Theorem in the case $\dim M = 1$ will be obvious.

f will be constructed as follows. If $f_0$ is a diffeomorphism which is structurally stable in strong sense and has a periodic point p that is a sink or source, then f will be obtained from $f_0$ by isotopically replacing $f_0$ on a small neighborhood of p.

Let $f_0$ be a time-one map of the flow of the vector field Y obtained by Theorem 2.1 of [5]. Then $f_0$ is a Morse-Smale diffeomorphism having a fixed point p which is a sink. By [3], $f_0$ is structurally stable in strong sense.

By replacing $f_0$ by an isotopy on a small neighborhood U of p we obtain $f_1$ such that

(i) every point in a small closed ball neighborhood B in U, with center p, is a fixed point of $f_1$, and

(ii) for every x in U-B, $\lim_{k\to\infty} f_1^k(x)$ exists in $\partial B$.

Let $B_r$ be a closed ball in the euclidean space $\mathbb{R}^m$ of the same dimension as M, centered on the origin with radius r. Let $S_r = \partial B_r$, a (m-1)-sphere. After this , we regard B as a closed ball $B_{r_0}$ in $\mathbb{R}^m$, and p as the origin of $\mathbb{R}^m$.

To construct f we will define a vector field V on B. On a neighborhood of p, f will be the time-one map of the flow of V.

(1) Construction of V.

For this purpose we at first define a vector field X. Let

$$\varphi_0(r) = e^{-1/r^2} \sin\frac{1}{r} \quad , \quad r > 0 \ .$$

Take $r_1 \in \mathbb{R}_+$ such that $r_1 < r_0$, $\varphi_0'(r_1) > 0$, and

$$\frac{1}{2n\pi} < r_1 < \frac{1}{(2n-1)\pi} \text{ for a fixed } n \in \mathbb{Z}_+ . \tag{2.1}$$

Let $\alpha:[\alpha_1,\infty)\to\mathbb{R}$ be a $C^1$-function such that $\alpha(r)<0$ and $\alpha'(r)<0$ for every $r\in[r_1,\infty)$, and that the function defined by

$$\varphi(r)=\begin{cases}0 & \text{if} \quad r=0\\ \varphi_0(r) & \text{if} \quad 0<r<r_1\\ \alpha(r) & \text{if} \quad r_1<r\end{cases}$$

is $C^1$.

Define a vector field X on B by

$$X_x=\begin{cases}\varphi(\|x\|)\dfrac{x}{\|x\|} & \text{if} \quad x\neq 0\\[2ex] 0 & \text{if} \quad x=0\end{cases}$$

Here, $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^m$.

We show that X is $C^1$. Let $X={}^t(x_1,\cdots,x_m)\in\mathbb{R}^m$ be a row vector, i.e. the transposition of $(x_1,\cdots,x_m)$. If $x\neq 0$

$$\frac{\partial}{\partial x_i}X_x=\frac{\partial}{\partial x_i}\left(\frac{\varphi(\|x\|)}{\|x\|}\right)x+\frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x$$

$$=\frac{\partial}{\partial x_i}\|x\|\,\frac{\varphi'(\|x\|)\|x\|-\varphi(\|x\|)}{\|x\|^2}x+\frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x$$

$$=\frac{x_i}{\|x\|}\,\frac{\varphi'(\|x\|)\|x\|-\varphi(\|x\|)}{\|x\|^2}x+\frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x$$

$$=x_i\left(\frac{\varphi'(\|x\|)}{\|x\|^2}-\frac{\varphi(\|x\|)}{\|x\|^3}\right)x+\frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x\ .$$

Hence, for $x\neq 0$

$$DX_x=\left(\frac{\varphi'(\|x\|)}{\|x\|^2}-\frac{\varphi(\|x\|)}{\|x\|^3}\right)x\cdot{}^tx+\frac{\varphi(\|x\|)}{\|x\|}I\ ,$$

where $DX_x$ is the Jacobian matrix, and I is the unit matrix.

For a matrix $A=(a_1,\cdots,a_m)$ with row vectors $a_1,\cdots,a_m$, we define the norm of A by

$$\|A\| = \max_j \|a_j\| .$$

Then,

$$\|DX_x\| \leqq \left|\frac{\varphi'(\|x\|)}{\|x\|^2} - \frac{\varphi(\|x\|)}{\|x\|^3}\right| \cdot \|x\|^2 + \left|\frac{\varphi(\|x\|)}{\|x\|}\right| .$$

$DX_0 = 0$ since $\varphi'(0)=0$. Therefore, since $\varphi$ is $C^1$, X is a $C^1$-vector field.

Next, we difine a vector field Y on B. Let $\mu:[0,\infty)\to[0,\infty)$ be a $C^1$-function such that

$$\begin{cases} \mu \geqq 0 , & \text{and} \\ \mu(r)=0 \text{ and } \mu'(r)=0 & \text{if } r=0 \text{ or } r \geqq r_1 . \end{cases}$$

Let C be a $C^1$-vector field, on the unit sphere $S^{m-1}$, such that C has two singular points $p_+$ and $p_-$, where $p_+$ is a source at the north pole and $p_-$ is a sink at the south pole, and such that every other trajectory of C goes out of $p_+$ and into $p_-$. Then Y is defined by

$$Y_x = \begin{cases} \mu(\|x\|)\ C_{x/\|x\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 . \end{cases}$$

For the calculation of the derivative of $Y_x$, we take a $C^1$-extension $\tilde{C}: U(S^{m-1}) \to \mathbb{R}^m$ of $C: S^{m-1} \to \mathbb{R}^m$, where $U(S^{m-1})$ is a neighborhood of $S^{m-1}$ in $\mathbb{R}^m$. Then, for $x\neq 0$, we have

$$\mu(\|x\|)\ C_{x/\|x\|} = \mu(\|x\|)\ \tilde{C}_{x/\|x\|} .$$

Let $e_i$ be the i-th row vector of the unit matrix I. Let $y = \frac{x}{\|x\|}$, and let D be the notation of the derivative of variable x. Since

$$\frac{\partial}{\partial x_i}\frac{x}{\|x\|} = -\frac{x_i}{\|x\|^3}x + \frac{1}{\|x\|}e_i \ , \quad \text{and}$$

$$DY_x = D\mu(\|x\|)\cdot\tilde{C}_y + \mu(\|x\|)\cdot D\tilde{C}_y\cdot D(\frac{x}{\|x\|}) \ ,$$

we have

$$\frac{\partial}{\partial x_i}Y_x = \frac{\partial}{\partial x_i}(\mu(\|x\|))\tilde{C}_y + \mu(\|x\|)\cdot D\tilde{C}_y\cdot\frac{\partial}{\partial x_i}\frac{x}{\|x\|}$$

$$= \frac{x_i}{\|x\|}\mu'(\|x\|)\tilde{C}_y + \mu(\|x\|)\cdot D\tilde{C}_y\cdot(-\frac{x_i}{\|x\|^3}x + \frac{1}{\|x\|}e_i).$$

Consequently, if $x \neq 0$ then

$$DY_x = \mu'(\|x\|)\tilde{C}_y\cdot{}^t y + \mu(\|x\|)\cdot D\tilde{C}\cdot(-\frac{1}{\|x\|^3}x\cdot{}^t x + \frac{1}{\|x\|}I).$$

Since $\mu(0) = \mu'(0) = 0$ we have $DY_0 = 0$. Therefore Y is a $C^1$-vector field.

The $C^1$-vector field V on B is defined by

$$V_x = X_x + Y_x \ .$$

Fig.4 shows the orbit structure of V. Here, we denote $B(k) = B_{1/k\pi}$ and $S(k) = \partial B(k)$. Every singular point of V is hyperbolic except p.

(2) Construction of f.

Let $\Psi_1 : B \to B$ be the time one map of the flow $\Psi$ of V. $\Psi_1$ is a $C^1$-diffeomorphism such that $B - \Psi_1(B)$ is an annulus which is diffeomorphic to $\partial B\times[0,1)$. Every fixed point of $\Psi_1$

is hyperbolic except p. The property (ii) of $f_1$ and the orbit structure of V enable us to obtain a diffeomorphism $f : M \to M$ satisfying the following property ;

(i) $f|B = \Psi_1$ ,

(ii) $f|(M-U) = f_1(M-U)$ ,

(iii) if $x \in U-B$ then $\lim_{k\to\infty} f^k(x)$ is the north pole or the south pole of $S(2n)$.

Moreover, $f|(M-B)$ is obtained from $f_1$ by an isotopy supported by U. Since $\Psi_1$ is isotopic to $i_B = f_1|B$ by the isotopy $\Psi_t$, $t \in [0,1]$, f is isotopic to $f_1$ by an isotopy supported by U.

In §§4,5 it is proved that f possesses the desired properties (a),(b) of Theorem.

## §4. Proof of tolerance-stability of f in $\mathrm{Diff}^1(M)$.

Let sufficiently small $\varepsilon > 0$ be given.

Lemma. There is a diffeomorphism $h : M \to M$ such that (i) $h = \text{identity}$ on $M-B_{\varepsilon/4}$, and (ii) $f_\varepsilon = hf$ is structurally stable in strong sense.

Proof. We may assume

(4.1) $$\frac{\varepsilon}{3} < r_1 .$$

Let $\ell$ be a sufficiently large integer satisfying the following unequalities.

(4.2) $$\frac{1}{2\ell\pi} + e^{-(\ell\pi)^2} < \frac{1}{(2\ell-1)\pi} < \frac{\varepsilon}{4} .$$

Put $\frac{1}{2\ell\pi} + e^{-(\ell\pi)^2} = r_2$ . Define a disconnected function $\eta_0 : (0, r_2) \to \mathbb{R}_+$ by

$$\eta_0(r) = \begin{cases} r - e^{-(k\pi)^2} & \text{if } \frac{1}{(k+1)\pi} < r \leqq \frac{1}{k\pi} , \\ r - e^{-4(\ell\pi)^2} & \text{if } \frac{1}{2\ell\pi} < r \leqq r_2 , \end{cases}$$

where $k = 2\ell,\ 2\ell+1,\ 2\ell+3, \cdots$. Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a $C^1$-function satisfying

$$(4.3) \qquad \begin{cases} 0 \leqq \eta(r) \leqq r , \\ \eta(r) = r \quad \text{if } r > \frac{1}{(2\ell-1)\pi} , \\ \eta(r) < \eta_0(r) \quad \text{if } 0 < r \leqq r_2 , \\ \eta(0) = 0 , \\ \eta'(r) > 0 \quad \text{for every } r \geqq 0 , \\ \eta'(0) < 1 . \end{cases}$$

In fact, $\eta$ exists. Especially we can fined $\eta$ such that $0 < \eta'(0) < 1$, since in a neighborhood of 0 the following properties hold.

$$(4.4) \qquad \eta_0(r) > r - e^{-(\frac{1}{r}-\pi)^2} ,$$

$$(4.5) \qquad \lim_{r\to 0} \frac{1}{r}(r - e^{-(\frac{1}{r}-\pi)^2}) = 1 .$$

Define $h : M \to M$ by

$$(4.6) \qquad h(x) = \begin{cases} \eta(\|x\|)\frac{x}{\|x\|} & \text{if } x \in B \\ x & \text{if } x \notin B . \end{cases}$$

Since $B = B_{r_0}$ and $r_1 < r_0$, the map h is well defined by (4.1), (4.2) and (4.3). h is a diffeomorphism. Define $f_\varepsilon$ by

$$f_\varepsilon(x) = hf(x).$$

By (4.3) $f_\varepsilon(x) = f(x)$ if $\|x\| \geqq 1/(2\ell-1)\pi$.

Next, we show

$$\|f_\varepsilon(x)\| < \|x\| \quad \text{if} \quad \|x\| < \frac{1}{(2\ell-1)\pi}. \tag{4.7}$$

Remember the definition of the vector field X, then we observe that $\|f(x)\| \leqq \|x\|$ when $\frac{1}{2k\pi} < \|x\| < \frac{1}{(2\ell-1)\pi}$. since $\eta(\|x\|) \leqq \|x\|$, it follows that

$$\|f_\varepsilon(x)\| < \|x\| \quad \text{if} \quad \frac{1}{2k\pi} < \|x\| < \frac{1}{(2k-1)\pi}, \quad k \geqq \ell. \tag{4.8}$$

Nest, let $\frac{1}{(2k+1)\pi} < \|x\| < \frac{1}{2k\pi}$. Let $\bar{\Psi}_t(x)$ be the flow of X, so that $\bar{\Psi}_0(x) = x$. Since $V_x = X_x + Y_x$ and $\|f(x)\| = \|\Psi_1(x)\| = \|\bar{\Psi}_1(x)\|$, we have

$$\|f(x)\| = \|x\| + \int_0^1 \varphi(\|\bar{\Psi}_t(x)\|)\,dt, \tag{4.9}$$

where $\varphi(r) = e^{-1/r^2} \sin\frac{1}{r}$ as before. $1/(2k+1)\pi \leqq \|x\| \leqq 1/2k\pi$ implies $0 \leqq \sin(1/\|x\|) \leqq 1$.

Hence,

$$\varphi(\|x\|) \leqq e^{-1/\|x\|^2} \leqq e^{-(2k\pi)^2}.$$

Therefore, by (4.9),

$$\|f(x)\| \leqq \|x\| + e^{-(2k\pi)^2}.$$

Using this and the defimition of $\eta_0$ we have

$$\begin{aligned}\|f_\varepsilon(x)\| &= \|hf(x)\| = \eta(\|f(x)\|)\\ &\leqq \eta(\|x\| + e^{-(2k\pi)^2})\\ &< \eta_0(\|x\| + e^{-(2k\pi)^2})\\ &< (\|x\| + e^{-(2k\pi)^2}) - e^{-(2k\pi)^2} = \|x\| .\end{aligned}$$

Hence,

$$(4,10) \qquad \|f_\varepsilon(x)\| < \|x\| \qquad \text{if} \quad \frac{1}{(2k+1)\pi} \leqq \|x\| \leqq \frac{1}{2k\pi} .$$

By (4.8) and (4.10) we have (4.7).

Hence $f_\varepsilon$ contracts to p in lnt $B(2\ell-1)$. We have $f_\varepsilon = f$ in $M-B_{1/(2\ell-1)\pi}$ by (4.3). By the definition of f, $f|(M-B_{1/(2\ell-1)\pi})$ is Morse-Smale and $\partial B_{1/(2\ell-1)\pi}$ is f-invariant. Therefore $f_\varepsilon$ is Morse-Smale. Since a Morse-Smale diffeomorphism is structurally stable in strong sense by [3] this completes the proof of Lemma.

Since $f_\varepsilon$ is structurally stable in strong sence there is a neighborhood $N_0$ of $f_\varepsilon$ in $\mathrm{Diff}^1(M)$ such that every element in $N_0$ is topologically $\varepsilon/24$-conjugate to $f_\varepsilon$.

Since h is a $C^1$-diffeomorphism the map $h_* : \mathrm{Diff}^1(M) \to \mathrm{Diff}^1(M)$ defined by $h_*(g) = hg$ is continuous [1, p.229, (B.8)]. Hence, for the neighborhood $N_0$ of $hf = f_\varepsilon$, there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ such that

$$g \in N \Rightarrow hg = g_\varepsilon \in N_0 .$$

Hereafter, let g is included in this N. Since $h = \text{identity}$ on $M - B_{\varepsilon/4}$ by (4.2), (4.3) and (4.6), we have

(4.11) $f_\varepsilon$ and $g_\varepsilon$ are topologically $\varepsilon/24$-conjugate

$f_\varepsilon = f$ and $g_\varepsilon = g$ in $M\text{-}B_{\varepsilon/4}$ .

There is a homeomorphism $h_g : M \to M$ such that

(4.12) $h_g g = f h_g$ and $d(h_g(x), x) < \varepsilon/24$, $\forall x$.

We may assume that $\varepsilon$ is so small as there is an integer k satisfying $3/\pi\varepsilon < k < 24/7\pi\varepsilon$ . Then we have

(4.13) $$\frac{\varepsilon}{4} + \frac{\varepsilon}{24} < \frac{1}{k\pi} < \frac{\varepsilon}{3} .$$

(4.1),(4.13) and the definition of f imply that $S_{1/k\pi}$ is f-invariant. Denote $S_f = S_{1/k\pi}$ . Since $S_f$ is contained in the complement of $B_{\varepsilon/4}$, (4.11) implies that $S_f$ is also $f_\varepsilon$-invariant. Since $f_\varepsilon$ and $g_\varepsilon$ are topologically $\varepsilon/24$-conjugate, (4.11) and (4.13) imply that $h_g(S_f)$ is contained in $M\text{-}B_{\varepsilon/4}$ and is both g and $g_\varepsilon$-invariant. Denote $h_g(S_f) = S_g$, $B_{1/k\pi} = B_f$ and $h_g(B_f) = B_g$. Since $\partial B_f = S_f$ and $\partial B_g = S_g$ we have

(4.14) $$\begin{cases} f_\varepsilon = f & \text{in } M - B_f , \\ g_\varepsilon = g & \text{in } M - B_g , \\ f\,|(M\text{-}B_f) \text{ and } g\,|(M\text{-}B_g) & \text{are topologically } \frac{\varepsilon}{24}\text{-conjugate.} \end{cases}$$

Precisely, the last part of (4.14) means that there is the commutative diagram

$$\begin{array}{ccc} (M - B_f) & \xrightarrow{f} & (M - B_f) \\ \downarrow h_g & & \downarrow h_g \\ (M - B_g) & \xrightarrow{g} & (M - B_g) \end{array}$$

and $d(h_g(x), x) < \varepsilon/24$ for $\forall x \in (M\text{-}B_f)$. (4.14) implies

$$(4.15)\quad \begin{cases} B_f \text{ is f-invariant ,} \\ B_y \text{ is g-invariant .} \end{cases}$$

For every g in N, we must show that f and g are orbit-$\varepsilon$-equivalent. First, let $O_f \subset M - B_f$. Then, $O_f$ is a $f_\varepsilon$-orbit $O_{f_\varepsilon}$. By (4.14), $h_g(O_f) = O_{g_\varepsilon}$ is contained in M-$B_g$ and $O_{g_\varepsilon}$ is a g-orbit $O_g$. Since $d(h_g, i_M) < \varepsilon/24$ then the conditions (a) and (b) of 1 in Definition (2.1) are satisfied in this case.

Next, let $O_f \subset B_f$. Take any orbit $O_g$ in $B_g$ (by using(4.15)). Then (a) and (b) of 1 in Definition (2.1) are satisfied. In fact, for any $x \in B_f$ and $y \in B_y$, by (4.13) we have

$$\begin{aligned} \|x - y\| &\leqq \|x\| + \|y\| \\ &\leqq \frac{1}{k\pi} + \left(\frac{1}{k\pi} + \frac{\varepsilon}{24}\right) \\ &< \frac{\varepsilon}{3} + \left(\frac{\varepsilon}{3} + \frac{\varepsilon}{24}\right) < \varepsilon \quad . \end{aligned}$$

Hence, the condition 1 in Definition (2.1) is satisfied. Similary we can show the condition 2 by dividing the case in $O_g \subset M - B_g$ and $O_g \subset B_g$. Therefore f is tolerance-stable in $\mathrm{Diff}^1(M)$.

§5. Proof of topological unstability in $\mathrm{Diff}^1(M)$.

Suppose that f is topologically stable in $\mathrm{Diff}^1(M)$. Then, for any $\varepsilon_1 > 0$ there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ such that for every g in N there is a continuous map $\tau : M \to M$ satisfing

(a) $d(\tau, i_M) < \frac{\varepsilon_1}{2}$ ,

(b) $\tau g = f\tau$ .

For the fixed interger n in (2.1), let

$$\varepsilon_1 = \frac{1}{2n\pi} \quad .$$

To introduce a contradiction, we take following g ;

$$g = hf \ ,$$

where h is a diffeomorphism defined by (4.6). But we must take g such that $g \in N$. By (4.4),(4.5) and the definition (4.3) of $\eta$ we can choose $\eta$, by taking $\ell$ sufficiently large, such that $|\eta(r) - r|$ and $|\eta'(r) - 1|$ are arbitrarily small. Hence we may assume that $g \in N$ and $\frac{1}{(2\ell-1)\pi} < \varepsilon_1$. Then any invariant closed subset of g, included in $B_{\varepsilon_1}$, contains at most two fixed points. (See Fig.6) Therfore, in $B_{\varepsilon_1}$ there is at most finite fixed point of g.

If y is a fixed point of f satisfying $\|y\| < \frac{\varepsilon_1}{2}$ then $\tau^{-1}(y)$ contains a fixed point of g. In fact, since $\tau g = f\tau$, $\tau^{-1}(y)$ is a g-invariant closed subset. By the condition (a) above, each x in $\tau^{-1}(y)$ satisfies

$$\begin{aligned} \|x\| &\leqq \|y\| + \|y - x\| \\ &= \|y\| + \|\tau x - x\| \\ &< \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1 \quad . \end{aligned}$$

Hence, for each fixed point y of f in $B_{\varepsilon_1/2}$, there is a fixed point x of g such that $\tau(x) = y$ and $x \in B_{\varepsilon_1}$. There are infinitely many fixed points of f in $B_{\varepsilon_1/2}$, but there are at most finite fixed points of g in $B_{\varepsilon_1}$. This is a contradiction. Therefore f is topologically unstable in $\mathrm{Diff}^1(M)$.

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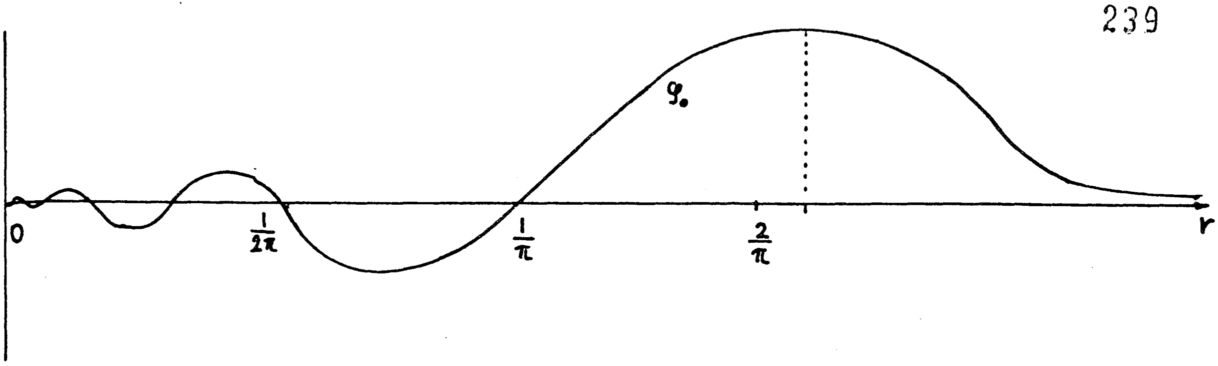


Fig. 1

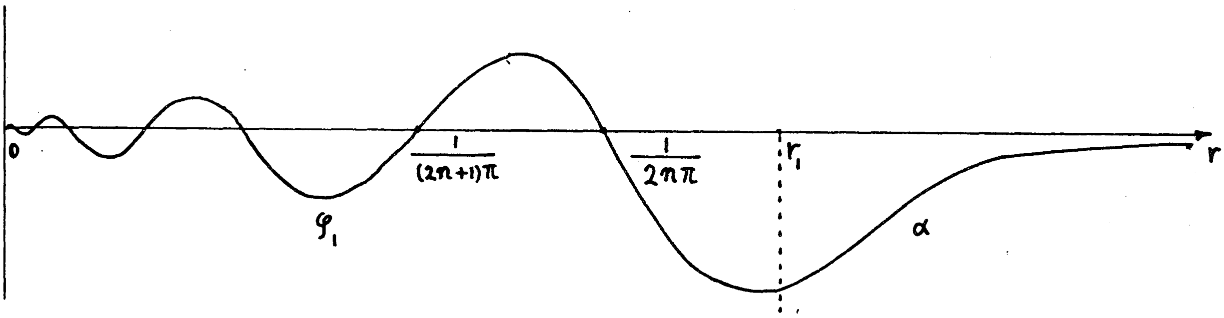


Fig. 2

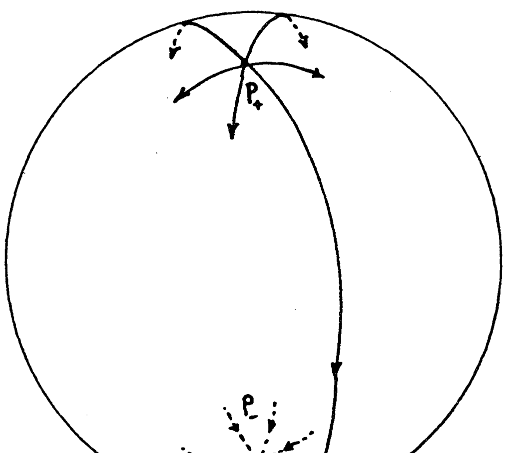


Fig. 3

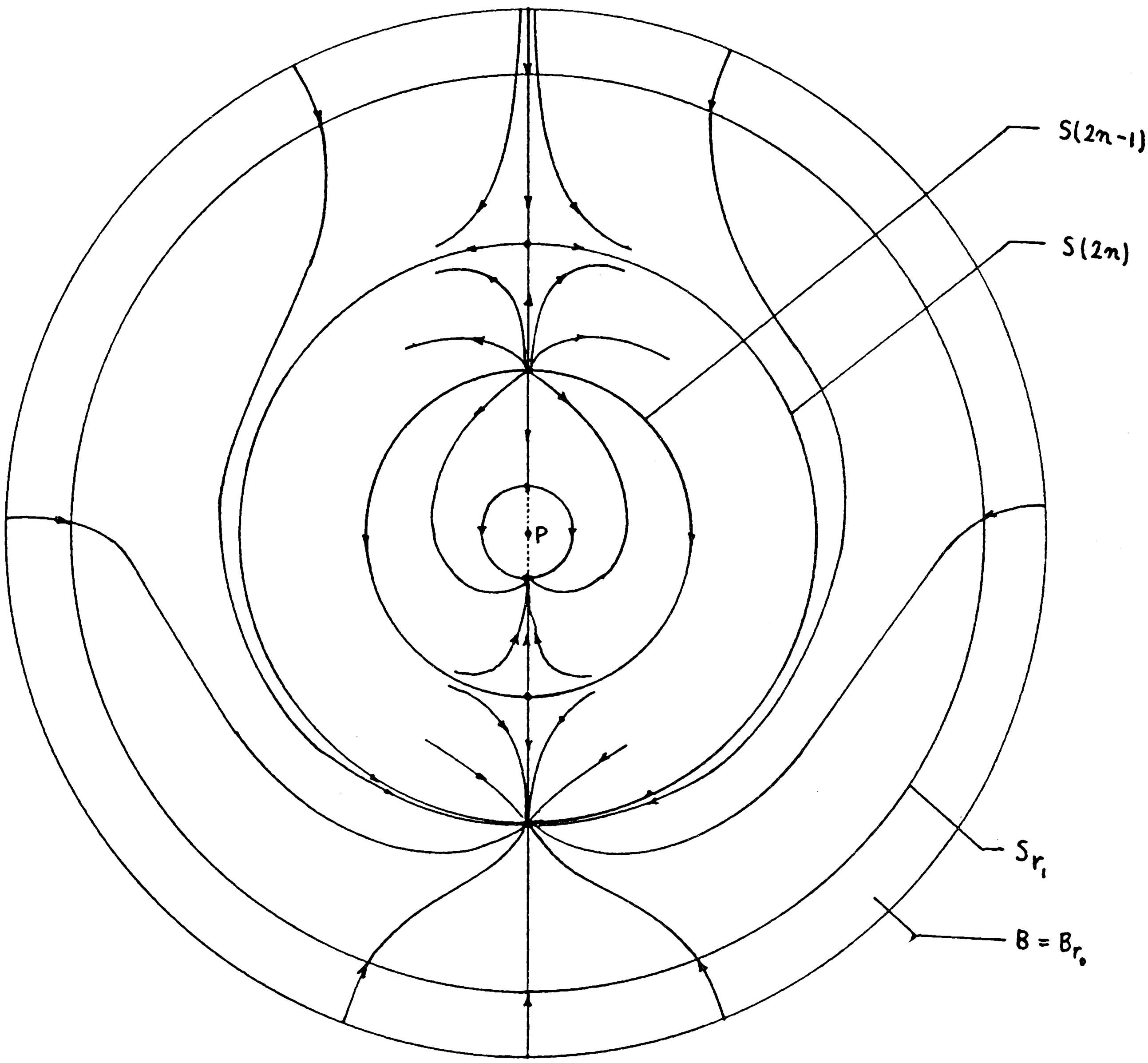


Fig. 4

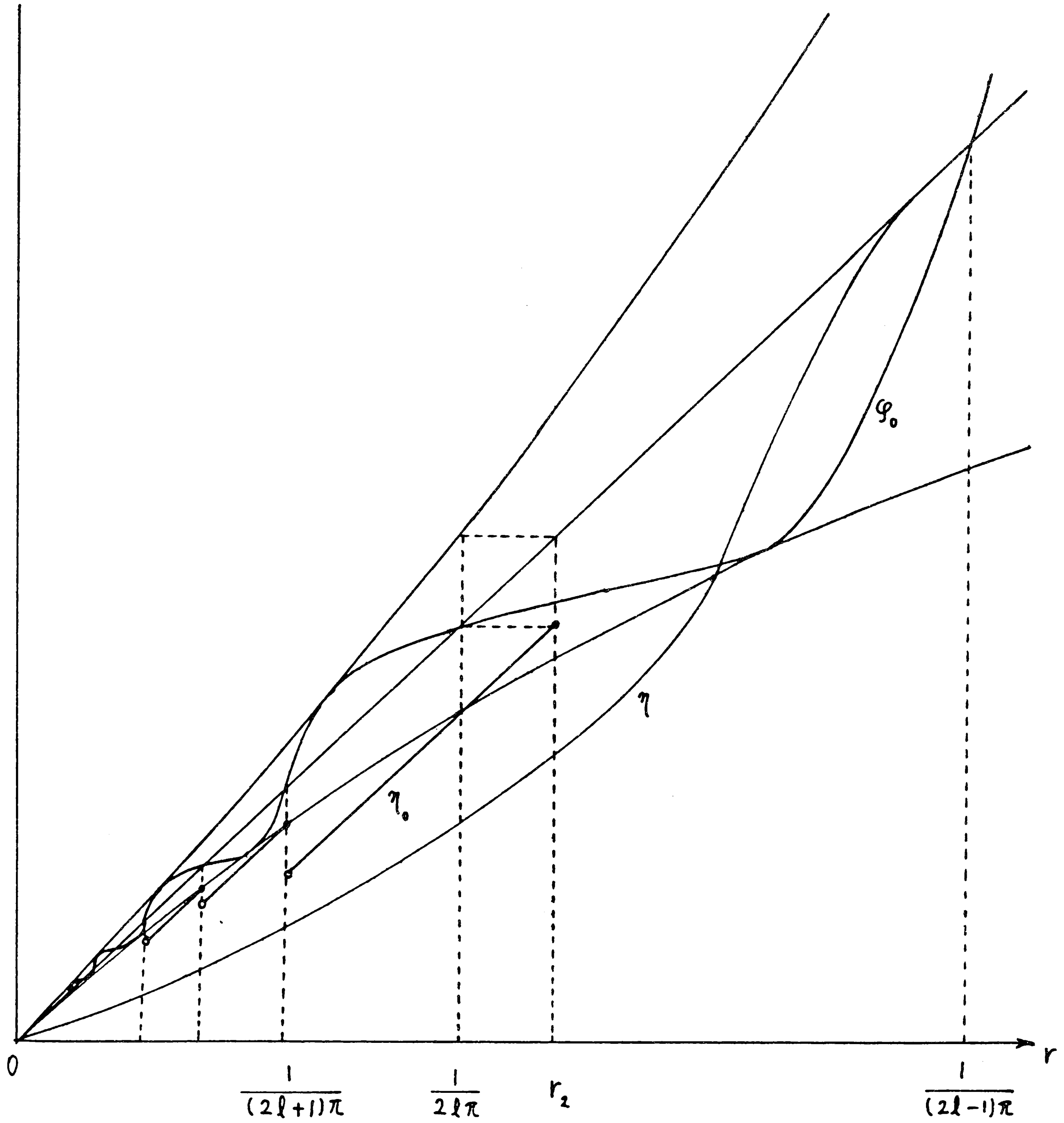


Fig. 5

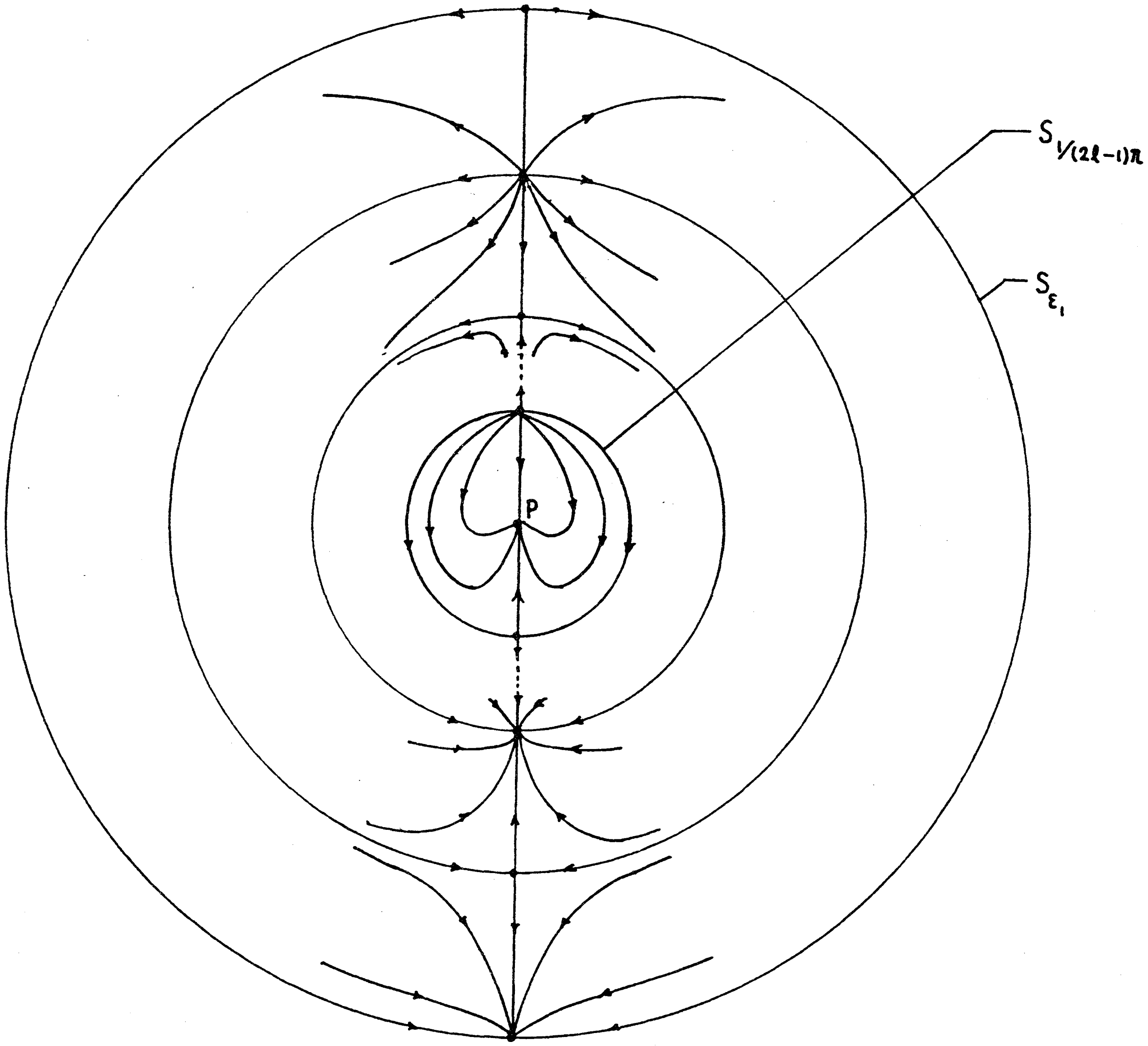


Fig. 6.